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Monterey, California

THESIS

A COMPUTATIONAL METHOD FOR WINGS OF ARBITRARY PLANFORM

by

Christopher Stephen Jones

December, 1984

Thesis Advisor: T.H. Gawain

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The computational method developed in this thesis permits the calculation of the aerodynamic performance of a wing of arbitrary planform. Both basic and additional lift are analyzed. This treatise is restricted to thin wings in steady, inviscid, incompressible flow. The method uses a grid system of control points over the wing semi-span. The circulation over the wing is considered variable with discrete values at

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A Computational Method
For Wings Of Arbitrary Planform

by

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Lieutenant Commander, United States Navy
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Submitted in partial fulfillment of the
requirements for the degree of

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December, 1984

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The computational method developed in this thesis permits the calculation of the aerodynamic performance of a wing of arbitrary planform. Both basic and additional lift are analyzed. This treatise is restricted to thin wings in steady, inviscid, incompressible flow. The method uses a grid system of control points over the wing semi-span. The circulation over the wing is considered variable with discrete values at the specified grid points. Finite difference equations are utilized to determine these discrete values. Control point indeterminacies are evaluated analytically. Matrix inversion is required for solution by the method presented. Details of the matrix technique are developed in Ref. 4. A brief summary of the principal computational relations is included. No numerical results are yet available but are expected during the next phase of this research.

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I. BACKGROUND

The overall research of which this thesis forms a part has a three-fold purpose. Firstly, it is intended to instruct the student about the fundamental equations that govern the aerodynamics of wings in incompressible flow. Secondly, it teaches the student how to convert the fundamental equations into a form suitable for numerical calculation on the digital computer, and to carry out the complex programming required for this purpose. Thirdly, once the first two objectives have been accomplished, the computer program itself provides a useful pedagogical and design tool for illustrating the effects of various wing design paramenters on the final aerodynamic performance of a wing.

Lcdr. J. L. Parks made a creditable start toward the first objective in his thesis [Ref. 1]. However, in the limited time available, he was unable to make significant progress toward the second or third objectives.

It was evident early in this present analysis that Parks' computational technique had some flaws. The expectation was that these flaws would be readily discovered and soon corrected and that objectives two and three would be rather quickly and easily completed. That has not been the case!

As the investigation actually developed, a whole series of obstacles was encountered and each was in turn eventually overcome. The progress was neither quick nor easy. In every case, it was found that the seeming obstacle was really attributable to some conceptual error. Once the error was found, the obstacle disappeared and the investigators gained a deeper insight. Some of the major obstacles that were encountered and eventually overcome are discussed in the following paragraphs.

Confusion about the proper type of computational grid to use, whether staggered or unstaggered, and the grid size necessary for accuracy was a major factor. An 8 by 8 unstaggered grid over the semi-span is now used, for good reason.

Another point of confusion was how to deal numerically with the singularity that occurs in the governing integral equation. One option is to avoid the problem by staggering the grid of field points with respect to the grid of control points. The other option, which was chosen for this investigation, resolves the indeterminacy by rigorous analysis and as a consequence allows the employment of a simple unstaggered mesh.

A third point of confusion concerned the evaluation of the partial derivatives of the circulation function. Should analytical or finite difference methods be used to represent these derivatives? It was found that analytical differentiation is incorrect and that finite differences must be used.

Still another confusing point concerned the validity of representing the circulation function by a Fourier series. While this procedure is widely advocated in the technical literature, it was found that a much simpler and clearer formulation can be obtained otherwise.

Boundary conditions also caused some confusion. Special conditions apply at the leading and trailing edges and at the wing tip and midspan. These quite complex boundary conditions have now been fully and rigorously analyzed and incorporated into the formulation of the problem.

It is evident from the foregoing discussion that this research has amounted to a major education in basic aerodynamics and in numerical methods. In these respects it has been a richly rewarding experience.

However, in view of the foregoing obstacles and problems, the time schedule has of course been greatly delayed from that which was initially anticipated. Thus, there are no final numerical results at this particular stage of the investigation. There is also no more time available to this investigator. Hence, this final aspect will have to be completed by some subsequent investigator. Nevertheless, what the present effort has produced, to aid any subsequent worker, is a very sophisticated and refined numerical method. This method has now evolved to the point that it can be confidently expected to produce a reliable and accurate final result.

II. INTRODUCTION

This analysis is a continuation of a previous study conducted by John L. Parks, Lieutenant Commander, United States Naval Reserve [Ref. 1].

The concept of a continuous vortex sheet of variable strength over the wing, trailing to infinity aft of the wing is utilized to determine the value of the circulation function at a finite number of control points on the wing. The vortex sheet strength is restricted by the Kutta condition at the trailing edge of the wing and by the requirement of no flow through the wing at the specified control points. The present analysis is restricted to steady, inviscid, incompressible flow about thin wings with straight or swept leading and trailing edges.

Wing aerodynamics involves two fundamental aspects. First, a desired basic lift distribution may be specified over the wing and the corresponding wing camber must be determined. Second, a flat plate wing may be specified with the resultant additional lift distribution to be calculated. Once the additional lift distribution is known, other pertinent aerodynamic parameters may be determined; notably, the slope of the wing lift curve and the location of the aerodynamic center. Also, when the basic and additional lift distributions are both known, the corresponding induced drag can readily be calculated.

This analysis develops a method to calculate the spanwise and chordwise pressure distributions over the wing which are necessary in designing the wing structure.

Although Parks' calculations did not achieve a satisfactory result, he pointed out areas of concern for possible errors and indicated a need for further investigation.

Where Parks used a staggered mesh between control points and field points on the wing to avoid the indeterminacy encountered whenever a control point and field point coincide, the present analysis resolves the indeterminacy and works with control points and field points superimposed upon each other. Also, the series solution employed by Parks to represent the circulation distribution has been discarded. Instead, this distribution is now represented more simply and directly by the values of the circulation at the field points themselves.

III. WING GEOMETRY AND COORDINATE TRANSFORMATION

A. WING GEOMETRY

The wing geometry considered in this development consists of a wing planform symmetric about midspan with straight leading and trailing edges containing no discontinuities, except at midspan where allowance is made for sweepback. The wing has no control surfaces. A typical wing is pictured in Fig. 1. with the geometric coordinates labeled. The wing is completely specified by three factors, the aspect ratio, taper, and the leading edge sweep.

All coordinates on the wing are normalized to a semi-span length of one unit. Sweepback at the wing tip is Lamda(λ). Eta(η) is the normalized spanwise coordinate. Parameter Tau(τ) defines wing taper. Xi(ξ) is the normalized chordwise distance from the leading edge. Sigma(σ) is a sign parameter, +1 for the right wing and -1 for the left wing. The mean chord($\bar{c}$) is a function of aspect ratio(AR), that is, $\bar{c} = 2/AR$. This analysis involves two coordinate transformations. The transformations proceed from the (x,y)-plane to the non-dimensional (ξ,η)-plane, then to the (ϕ,θ) angular coordinate plane.

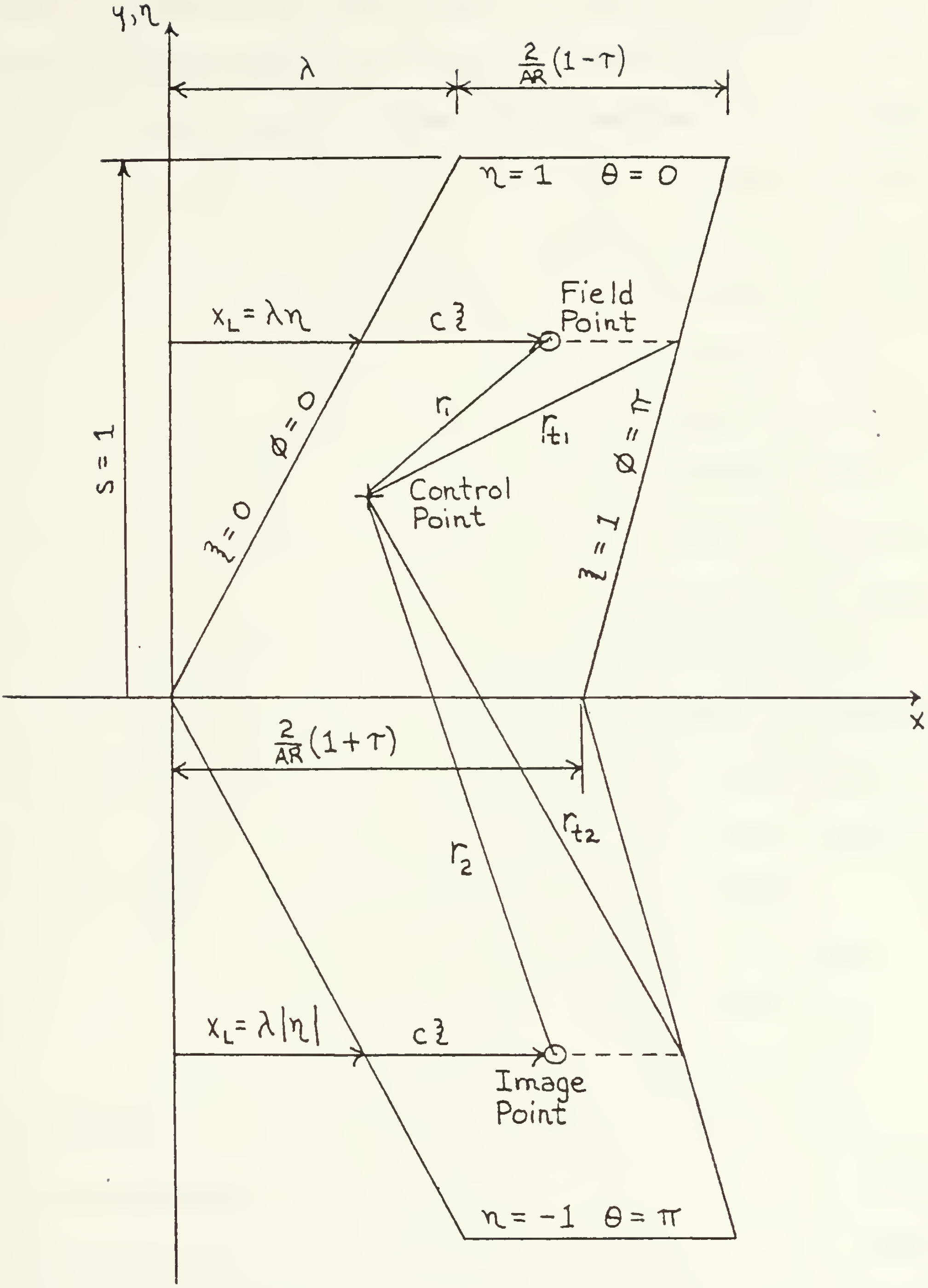


Figure 1. Wing Coordinates

B. NON-DIMENSIONAL COORDINATES

The first change of variables involves a transformation from the (x,y)-plane into the (ξ,η)-plane. In the following relationships, non-subscripted variables refer to field points, while variables with subscript 'p' refer to control points.

$$y = \eta \; ; \; dy = d\eta \tag{3.1}$$

$$y_p = \eta_p > 0 \tag{3.2}$$

$$x_L = \lambda\sigma\eta \tag{3.3}$$

$$x_{Lp} = \lambda\eta_p \tag{3.4}$$

$$c = (2/AR)\,[(1+\tau) - 2\tau\sigma\eta] \tag{3.5}$$

$$c_p = (2/AR)\,[(1+\tau) - 2\tau\eta_p] \tag{3.6}$$

$$x = x_L + c\xi \tag{3.7}$$

$$x_p = x_{Lp} + c_p\xi_p \tag{3.8}$$

$$dx_L = \lambda\sigma d\eta \tag{3.9}$$

$$dc = -(2/AR)2\tau\sigma d\eta \tag{3.10}$$

$$dx = dx_L + cd\xi + \xi dc \tag{3.11}$$

$$= \lambda\sigma d\eta + cd\xi + \xi[-(2/AR)2\tau\sigma d\eta] \tag{3.12}$$

$$\mu = [\lambda - (2/AR)2\tau\xi] \tag{3.13}$$

$$dx = cd\xi + \mu\sigma d\eta \tag{3.14}$$

$$\tau = (1 - \text{taper ratio})/(1 + \text{taper ratio}) \tag{3.15}$$

This transformation simplifies the algebra of the problem significantly. The chordwise dimension now varies from 0 at the leading edge to +1 at the trailing edge, while the spanwise dimension varies from -1 at the left wing tip to +1 at the right wing tip.

C. ANGULAR COORDINATES

It is desirable to transform the wing planform from the (ξ,η)-plane into the (ϕ,θ)-plane. This is accomplished using the relationships

$$\xi = 0.5 * (1 - \cos\phi); \quad d\xi = 0.5 \sin\phi \, d\phi \qquad (3.16)$$

$$\eta = \cos\theta; \quad d\eta = -\sin\theta \, d\theta \qquad (3.17)$$

From equation (3.16), as ξ varies from zero at the leading edge to 1 at the trailing edge, ϕ will vary from zero to π, respectively. From equation (3.17), as η varies from -1 at the left wing tip to +1 at the right wing tip, θ will vary from π to zero, respectively. These limits are important in subsequently developed integral equations. An added bonus of this transformation is that it results in mesh points near the leading and trailing edges and wing tips to be closely spaced, allowing finer resolution in these critical areas of analysis. Fig. 2 shows the relationship between the linear coordinates (ξ,η) and the angular coordinates (ϕ,θ).

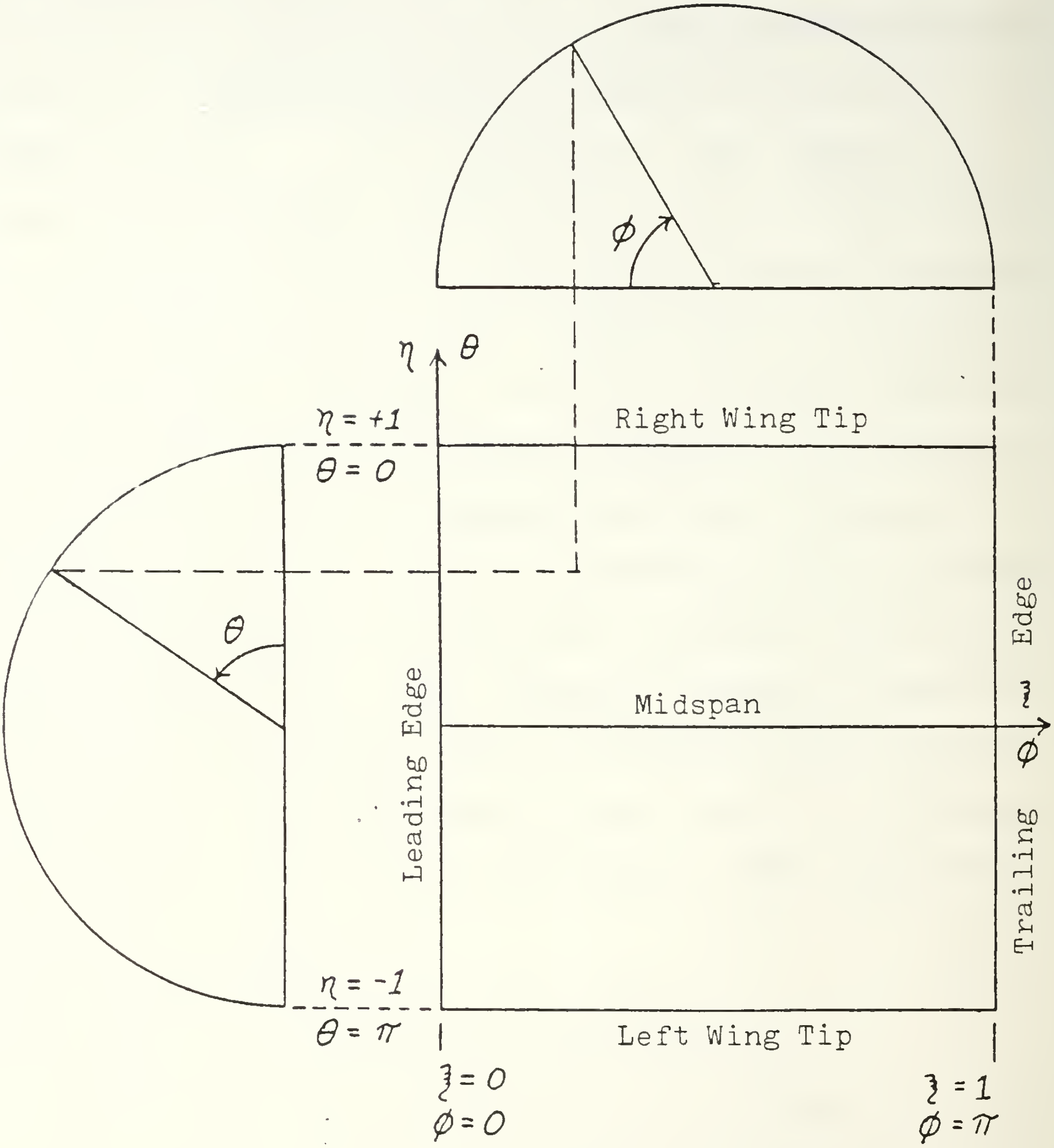


Figure 2. Linear Coordinates (ξ,η) versus Angular Coordinates (ϕ,θ)

IV. GOVERNING EQUATIONS

The vertical velocity induced at any particular control point consists of that contribution made by the vorticity distribution over the wing planform plus the contribution made by the trailing vortices behind the wing. Parks [Ref. 1] provides a quite detailed derivation of the two general equations for induced velocity. Both equations may also be found in many other aeronautical references such as Bertin and Smith [Ref. 2] and Kuethe and Chow [Ref. 3]. The following form of these equations was utilized in this analysis

$$w_p' = \frac{1}{4\pi}\iint_S \frac{1}{r^3}\left\{(x-x_p)\frac{\partial\hat{\Gamma}}{\partial x} + (y-y_p)\frac{\partial\hat{\Gamma}}{\partial y}\right\} dx\,dy \qquad (4.1)$$

$$w_p'' = \frac{1}{4\pi}\int \frac{1}{(y-y_p)}\left\{1 - \frac{(x_t-x_p)}{r_t}\right\}\left(\frac{d\hat{\Gamma}}{dy}\right)_t dy \qquad (4.2)$$

where

$$r^2 = (x - x_p)^2 + (y - y_p)^2 \qquad (4.3)$$

and r_t^2 is obtained by replacing x with x_t in equation (4.3).

Equation (4.1) is basically the same as Park's equation (3.6) where w_p' is the velocity induced at a control point due to the vorticity distribution over the wing surface. Equation (4.2) is the same as Park's equation (3.8) where w_p'' is the velocity induced at a control point due to the trailing vorticity behind the wing. Subscript 't' here refers to values along the trailing edge of the wing. It is desirable to transform these equations to equivalent forms in the angular coordinate system.

A. CHANGE OF Γ TO WING COORDINATES

Circulation has the dimensions of velocity times distance. Consider two separate circulation functions, one non-dimensional with respect to semi-span and free stream velocity, and one non-dimensional with respect to mean chord and free stream velocity. The following relationships can then be established

$$V_\infty s\,\hat{\Gamma}(x,y) = V_\infty \bar{c}\,\tilde{\Gamma}(\xi,\eta); \quad \bar{c}/s = 2/AR \tag{4.4}$$

$$\hat{\Gamma}(x,y) = (2/AR)\,\tilde{\Gamma}(\xi,\eta) \tag{4.5}$$

$$\left(\frac{\partial\hat{\Gamma}}{\partial x}\right)dx + \left(\frac{\partial\hat{\Gamma}}{\partial y}\right)dy = \frac{2}{AR}\left\{\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right)d\xi + \left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right)d\eta\right\} \tag{4.6}$$

Substituting equations (3.1) and (3.14) into (4.6) gives

$$\left(\frac{\partial\hat{\Gamma}}{\partial x}\right)\left[c\,d\xi + \mu\sigma\,d\eta\right] + \left(\frac{\partial\hat{\Gamma}}{\partial y}\right)d\eta = \frac{2}{AR}\left\{\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right)d\xi + \left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right)d\eta\right\} \tag{4.7}$$

Regrouping terms

$$\left[\left(\frac{\partial\hat{\Gamma}}{\partial x}\right)c - \frac{2}{AR}\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right)\right]d\xi + \left[\left(\frac{\partial\hat{\Gamma}}{\partial y}\right) - \frac{2}{AR}\left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right) + \left(\frac{\partial\hat{\Gamma}}{\partial x}\right)\mu\sigma\right]d\eta = 0 \tag{4.8}$$

The coefficients of $d\xi$ and of $d\eta$ must separately vanish, giving two equations which reduce to

$$\left(\frac{\partial\hat{\Gamma}}{\partial x}\right) = \frac{2}{AR}\,\frac{1}{c}\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right) \qquad (4.9)$$

$$\left(\frac{\partial\hat{\Gamma}}{\partial y}\right) = \frac{2}{AR}\left[\left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right) - \frac{\mu\sigma}{c}\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right)\right] \qquad (4.10)$$

Now, the following expression that occurs in equation (4.1)

$$(x - x_p)\left(\frac{\partial\hat{\Gamma}}{\partial x}\right) + (y - y_p)\left(\frac{\partial\hat{\Gamma}}{\partial y}\right) \qquad (4.11)$$

can be expressed in terms of ξ and η as follows

$$\frac{2}{AR}\left\{(x-x_p)\frac{1}{c}\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right) + (\eta-\eta_p)\left[\left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right) - \frac{\mu\sigma}{c}\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right)\right]\right\} \qquad (4.12)$$

Regrouping terms in equation (4.12) and setting

$$F = (x - x_p) - (\eta - \eta_p)\mu\sigma \qquad (4.13)$$

$$dx\ dy = c\ d\xi\ d\eta \qquad (4.14)$$

Equation (4.1) may now be written in the following form

$$w_p' = \frac{1}{4\pi}\left(\frac{2}{AR}\right)\int_{-1}^{+1}\int_{0}^{1}\frac{1}{r^3}\left\{F\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right) + c(\eta-\eta_p)\left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right)\right\}d\xi\,d\eta \qquad (4.15)$$

B. CHANGE OF Γ TO ANGULAR COORDINATES

Utilizing equations (3.16) and (3.17) together with a non-dimensional circulation function in the (ϕ,θ)-plane the following transformation can be made

$$\tilde{\Gamma}(\xi,\eta) = \Gamma(\phi,\theta) \tag{4.16}$$

$$\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right)d\xi + \left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right)d\eta = \left(\frac{\partial\Gamma}{\partial\phi}\right)d\phi + \left(\frac{\partial\Gamma}{\partial\theta}\right)d\theta \tag{4.17}$$

$$\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right)\frac{\sin\phi}{2}\,d\phi + \left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right)(-\sin\theta\ d\theta) = \left(\frac{\partial\Gamma}{\partial\phi}\right)d\phi + \left(\frac{\partial\Gamma}{\partial\theta}\right)d\theta \tag{4.18}$$

from which it follows that

$$\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right) = \frac{2}{\sin\phi}\left(\frac{\partial\Gamma}{\partial\phi}\right) \tag{4.19}$$

$$\left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right) = -\frac{1}{\sin\theta}\left(\frac{\partial\Gamma}{\partial\theta}\right) \tag{4.20}$$

and

$$\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right)d\xi d\eta = \frac{2}{\sin\phi}\left(\frac{\partial\Gamma}{\partial\phi}\right)\left(\frac{\sin\phi}{2}d\phi\right)\left(-\sin\theta d\theta\right) = -\sin\theta\left(\frac{\partial\Gamma}{\partial\phi}\right)d\phi\, d\theta \tag{4.21}$$

$$\left(\frac{\partial\tilde{\Gamma}}{\partial\eta}\right)d\xi d\eta = -\frac{1}{\sin\theta}\left(\frac{\partial\Gamma}{\partial\theta}\right)\left(\frac{\sin\phi}{2}d\phi\right)\left(-\sin\theta d\theta\right) = +\frac{\sin\phi}{2}\left(\frac{\partial\Gamma}{\partial\theta}\right)d\phi d\theta \tag{4.22}$$

Equation (4.15) now becomes

$$w_p{}' = \frac{1}{4\pi}\left(\frac{2}{AR}\right)\int_{\theta=\pi}^{0}\int_{\phi=0}^{\pi}\frac{1}{r^3}\left\{-F\sin\theta\left(\frac{\partial\Gamma}{\partial\phi}\right) + C(\eta-\eta_p)\frac{\sin\phi}{2}\left(\frac{\partial\Gamma}{\partial\theta}\right)\right\}d\phi d\theta \tag{4.23}$$

Upon reversing the limits of integration and reversing the algebraic sign of the integrand, equation (4.23) can be written as

$$w_p' = \frac{1}{4\pi}\left(\frac{2}{AR}\right)\int_0^{\pi}\int_0^{\pi}\frac{1}{r^3}\left\{+F\sin\theta\left(\frac{\partial\Gamma}{\partial\phi}\right)-c(\eta-\eta_p)\left(\frac{\partial\Gamma}{\partial\theta}\right)\right\}d\phi\, d\theta \qquad (4.24)$$

Equation (4.24) is the governing equation for the induced velocity at each control point due to the summation of the effects of every field point on the wing planform. Note that all factors inside the integral are a function of wing planform geometry except for the two partial derivatives of the circulation function. Also note that equation (4.24) becomes indeterminate when $x = x_p$ and $\eta = \eta_p$. This condition occurs when the control point and the field point coincide and is discussed later.

C. TRAILING VORTICITY TRANSFORMATION

Many of equations (3.1) through (3.14) are identical at the trailing edge and simply require the subscript 't' to be added to the appropriate variable. However, since ξ is constant at +1 along the trailing edge a few of the equations change slightly and are rewritten here for clarity

(3.7) becomes $x_t = x_L + c$ (4.25)

(3.8) becomes $x_p = x_{Lp} + c_p$ (4.26)

(3.12) becomes $dx_t = [\lambda - (2/AR)2\Upsilon]\sigma d\eta$ (4.27)

(3.14) becomes $dx_t = \mu\sigma d\eta$ (4.28)

The circulation along the trailing edge of the wing is a function of the spanwise coordinate η only. Consider again two non-dimensional circulation functions. The development is similar to that presented above for the circulation over the wing and results in the following

$$V_\infty s \hat{\Gamma}_t(y) = V_\infty \bar{c} \tilde{\Gamma}_t(\eta) \text{ and } \bar{c}/s = 2/AR \tag{4.29}$$

$$\hat{\Gamma}_t(y) = (2/AR)\tilde{\Gamma}_t(\eta) \tag{4.30}$$

$$\left(\frac{d\hat{\Gamma}_t}{dy}\right)dy = \frac{2}{AR}\left(\frac{d\tilde{\Gamma}_t}{d\eta}\right)d\eta \text{ and } dy = d\eta \tag{4.31}$$

$$\left(\frac{d\hat{\Gamma}_t}{dy}\right) = \frac{2}{AR}\left(\frac{d\tilde{\Gamma}_t}{d\eta}\right) \tag{4.32}$$

$$r_t^2 = (x_t - x_p)^2 + (\eta - \eta_p)^2 \tag{4.33}$$

$$G = 1 - [(x_t - x_p) / r_t] \tag{4.34}$$

Equation (4.2) may now be written as

$$w_p'' = \frac{1}{4\pi}\left(\frac{2}{AR}\right)\int_{-1}^{+1} \frac{G}{(\eta - \eta_p)}\left(\frac{d\tilde{\Gamma}_t}{d\eta}\right)d\eta \tag{4.35}$$

Now, transforming to angular coordinates

$$\tilde{\Gamma}_t(\eta) = \Gamma_t(\theta) \tag{4.36}$$

$$\left(\frac{d\tilde{\Gamma}_t}{d\eta}\right)d\eta = \left(\frac{d\Gamma_t}{d\theta}\right)d\theta \tag{4.37}$$

Equation (4.35) now becomes

$$w_p'' = \frac{1}{4\pi}\left(\frac{2}{AR}\right)\int_{\pi}^{0} \frac{G}{(\eta - \eta_p)}\left(\frac{d\Gamma_t}{d\theta}\right)d\theta \tag{4.38}$$

Again, reversing the limits of integration and the algebraic sign of the integrand, the integral equation for the induced velocity due to the trailing vorticity becomes

$$w_p'' = -\frac{1}{4\pi}\left(\frac{2}{AR}\right)\int_0^{\pi}\frac{G}{(\eta-\eta_p)}\left(\frac{d\Gamma_t}{d\theta}\right)d\theta \qquad (4.39)$$

D. TOTAL VORTICITY EFFECT

The total induced velocity at a particular control point is the summation of the effect produced by the vorticity over the wing and the effect produced by the trailing vorticity. This means that for each control point, the right hand side of equation (4.24) must be evaluated at every field point over the entire wing surface and summed. This sum will give the effect produced by the vorticity over the wing. Also, for each control point, the right hand side of equation (4.39) must be evaluated at each station along the entire trailing edge of the wing and summed. This sum will give the effect produced by the trailing vorticity. When both of these are added, the total vorticity effect at that particular control point is evaluated. This may be expressed as a combination of equations (4.24) and (4.39) as follows, where w_p is the total induced velocity

$$w_p = w_p' + w_p'' \qquad (4.40)$$

V. GOVERNING EQUATION INDETERMINACIES

Equations (4.24) and (4.39) may be evaluated by several different methods. The relationship between the geometric placement of the control points and field points is perhaps one of the most important factors. Wing symmetry about midspan permits the analysis to utilize control points over only one semi-span of the wing. The effect of field points and trailing vorticity must include the entire wing planform; however, symmetry about midspan again permits utilization of numerical values over only one semi-span. The right semi-span is chosen for this analysis.

As mentioned before, equation (4.24) becomes indeterminate if the control point mesh and the field point mesh coincide. Equation (4.39) also becomes indeterminate when the spanwise coordinate of the trailing vorticity and the concerned control point coincide. Note that G in equation (4.34) approaches zero as the spanwise coordinate of the control point and the trailing vorticity converge. One method to avoid these indeterminate points is to stagger the different meshes so that they never coincide. Another method is to determine an analytical limit or principal value of the integral as the indeterminate point is approached. The second method is chosen for this analysis and is presented in the following paragraphs.

A. INDETERMINATE POINT OVER WING

When a control point and a field point coincide, the numerator and the denominator of equation (4.24) are both zero resulting in an indeterminacy. Since an indeterminacy can only occur at a control point, all occurrences are confined to the right semi-span. All geometric variables and sigma, the sign parameter, have a positive value on this section of the wing. Fig. 3 depicts an exploded view of a mesh point singularity. The discrete value of the circulation at the singular point must be determined. Recall that the circulation function itself is continuous over the wing; therefore, it is continuous in the proximity of the singular point. Variables 'u' and 'v' are introduced to represent small deviations in ϕ and θ, in the vicinity of the singularity. If $\Delta R'$ represents the contribution of the singular element to the induced velocity at the singular point, the following relationship is established

$$\Delta R' = \int_{-\delta}^{+\delta}\int_{-\epsilon}^{+\epsilon} \Psi \, du \, dv \tag{5.1}$$

where symbol Ψ is an abbreviation that stands for the integrand of equation (4.24). Now, let

$$u = (\phi - \phi_p) \tag{5.2}$$

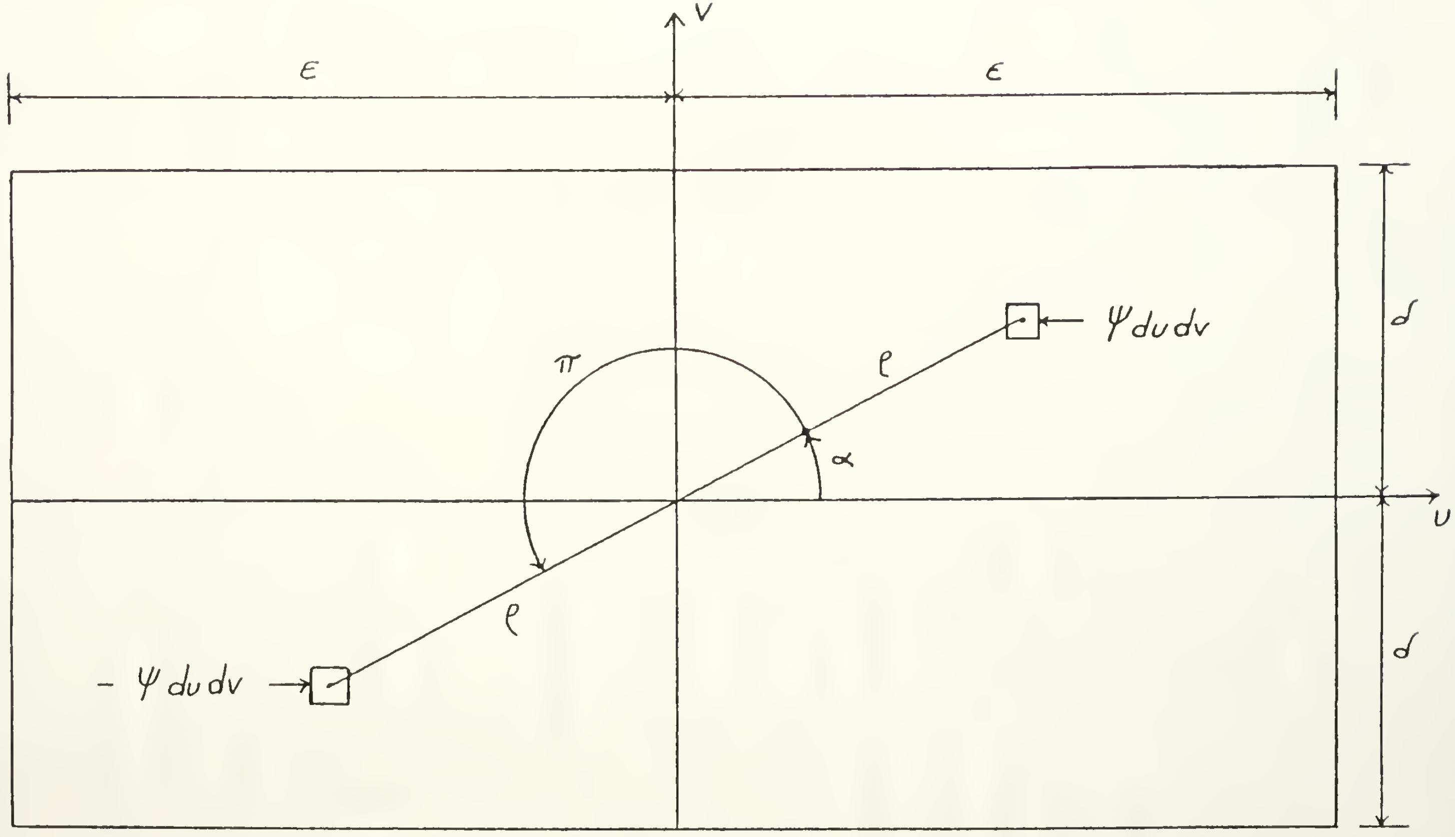


Figure 3. Field Element at Singular Point

$$v = (\theta - \theta_p) \tag{5.3}$$

$$\psi(u,v) = F'/r^3 \tag{5.4}$$

where F' represents the expression in brackets in equation (4.24).

An analysis of equation (4.24) in the immediate vicinity of the singularity results in the following

$$x = \lambda\eta + (2/AR)[(1 + \tau) - 2\tau\eta]\xi \tag{5.5}$$

$$x_p = \lambda\eta_p + (2/AR)[(1 + \tau) - 2\tau\eta_p]\xi_p \tag{5.6}$$

$$(x-x_p) = \lambda(\eta-\eta_p)+(2/AR)[(1+\tau)(\xi-\xi_p)-2\tau(\eta\xi-\eta_p\xi_p)] \tag{5.7}$$

$$\eta = \cos\theta = \cos(\theta_p+v) = \cos\theta_p - \sin\theta_p\, v \tag{5.8}$$

$$\eta_p = \cos\theta_p \tag{5.9}$$

$$(\eta - \eta_p) = -\sin\theta_p\, v \tag{5.10}$$

$$\xi = 0.5(1 - \cos\phi) = 0.5[1 - \cos(\phi_p+u)]$$

$$= 0.5(1 - \cos\phi_p + \sin\phi_p\, u) \tag{5.11}$$

$$\xi_p = 0.5(1 - \cos\phi_p) \tag{5.12}$$

$$(\xi - \xi_p) = 0.5\sin\phi_p\, u \tag{5.13}$$

$$(\eta\xi - \eta_p\xi_p) = (\eta_p-\sin\theta_p\, v)(\xi_p+0.5\sin\phi_p\, u)-\eta_p\xi_p \tag{5.14}$$

$$\begin{aligned} &= \eta_p\xi_p + \eta_p\, 0.5\sin\phi_p\, u - \sin\theta_p\, v\, \xi_p \\ &\quad - \sin\theta_p\, v\, 0.5\sin\phi_p\, u - \eta_p\xi_p \end{aligned} \tag{5.15}$$

Simplifying and neglecting higher order terms in u^2, v^2, and uv,

$$(\eta\xi - \eta_p\xi_p) = [(\eta_p/2)\sin\phi_p]u + (-\sin\theta_p\ \xi_p)v + \ldots \quad (5.16)$$

Substituting equations (5.10), (5.13), and (5.16) into (5.7),

$$(x - x_p) = Au + Bv \qquad (5.17)$$

where A and B are known functions of λ, θ_p, ϕ_p, η_p, ξ_p, τ, and AR. The denominator of the integrand of equation (4.24), in the immediate vicinity of the singular point may now be expressed as

$$r^3 = [(Au + Bv)^2 + (-\sin\theta_p\ v)^2]^{1.5} \qquad (5.18)$$

Since θ_p is fixed, let $-\sin\theta_p = C$, then,

$$r^3 = [(Au + Bv)^2 + C^2v^2]^{1.5} \qquad (5.19)$$

Assume, by analogy with the above development, that the numerator of equation (4.24) may also be expressed as a function of u and v, namely

$$F' = Du + Ev \qquad (5.20)$$

Now, switching to polar coordinates with 'ρ' the polar radius and 'α' the angle, let $u = \rho\cos\alpha$ and $v = \rho\sin\alpha$. The effect of the singular field element must be determined. An integration over the entire element produces the required result. The integration covers a total angle of 2π and a polar radius varying from zero at the center of the element to its outer bounds. Equations (5.19) and (5.20) expressed in polar coordinates become

$$r^3 = [(A\cos\alpha - B\sin\alpha)^2 + C^2 \sin^2\alpha]^{1.5} \rho^3 \tag{5.21}$$

$$= J(\alpha) \rho^3 \tag{5.22}$$

where $J(\alpha)$ is an abbreviated form of the term in brackets.

$$F' = [D \cos\alpha + E \sin\alpha] \tag{5.23}$$

$$= H(\alpha) \rho \tag{5.24}$$

where $H(\alpha)$ is also an abbreviated form.

Note the polar symmetry of each of the above functions, that is

$$J(\alpha + \pi) = J(\alpha) \tag{5.25}$$

$$H(\alpha + \pi) = -H(\alpha) \tag{5.26}$$

Therefore, if $\Delta R'$ is the effect of the indeterminacy

$$\Delta R' = \iint \frac{H(\alpha)\rho}{G(\alpha)\rho^3} (\rho \, d\alpha \, d\rho) = 0 \tag{5.27}$$

The value of this integral is zero by reason of odd polar symmetry. This proves that the effect of the singular field element on the control point is zero and may be neglected.

B. INDETERMINATE POINT ALONG TRAILING EDGE

As η approaches η_p in equation (4.39), G approaches zero; therefore, the integrand is indeterminate. Note that $(x_t - x_p)$ does not approach zero, since x_p is the x-coordinate of the center of the concerned control point and x_t is the x-coordinate at the trailing edge of the wing. In the immediate vicinity of the singularity, $(\eta - \eta_p)$ becomes very very small, but remains finite; therefore, it can be represented as a small value times a constant. Let this constant be $(x_t - x_p)$: therefore, as $(\eta - \eta_p)$ approaches zero, the following relationships may be established

$$(\eta - \eta_p) = e*(x_t - x_p), \text{ where } e \ll 1 \qquad (5.28)$$

$$\begin{aligned} r_t^2 &= (x_t - x_p)^2 + (\eta - \eta_p)^2 \\ &= (x_t - x_p)^2(1 + e^2) \end{aligned} \qquad (5.29)$$

$$\begin{aligned} r_t &= (x_t - x_p)(1 + e^2)^{0.5} \\ &= (x_t - x_p)(1 + e^2/2 + \ldots \text{H.O.T.}) \end{aligned} \qquad (5.30)$$

The integrand of equation (4.39) may now be expressed as

$$\frac{1}{(\eta-\eta_p)}\left\{1 - \frac{x_t - x_p}{r_t}\right\} = \frac{1}{e(x_t - x_p)}\left\{1 - \frac{x_t - x_p}{(x_t - x_p)(1 + \frac{e^2}{2}}\right\} \qquad (5.31)$$

which can be reduced to

$$\frac{1}{(x_t - x_p)}\left(\frac{e}{1 + e^2/2}\right) \qquad (5.32)$$

Thus, in the limit, as η approaches η_p, e approaches zero, and the value of equation (5.32) becomes zero. This proves that the trailing vorticity contribution directly downstream of a control point makes no contribution to the velocity induced at the control point.

VI. PRELIMINARY PROBLEM ANALYSIS

Consider a given wing planform for which the three parameters aspect ratio, taper, and leading edge sweep are known values. The corresponding lift distribution and therefore the circulation distribution over this wing are quantities to be determined. All quantities in the integrands of equations (4.24) and (4.39) are known except for the partial derivatives of the circulation function.

A mesh size adequate to represent the desired lift distribution, while not requiring an excessive amount of numerical calculations, is desired. A mesh consisting of eight chordwise stations and sixteen stations across the entire wing span is arbitrarily selected and should produce the desired accuracy. As previously mentioned, wing symmetry about midspan allows all computations to be made with the numerical values on the right semi-span.

A. WING GRID POINTS

This analysis involves mapping points on the wing from the physical (x,y)-plane to corresponding points in the (ϕ,θ)-plane and vice versa. Fig. 4 depicts a uniform rectangular grid of points in the (ϕ,θ)-plane, constructed as follows

$$\phi_i = \Delta\phi(i-0.5); \quad \Delta\phi = 2\epsilon = \pi/8; \quad i = 1,2,\ldots,8 \qquad (6.1)$$

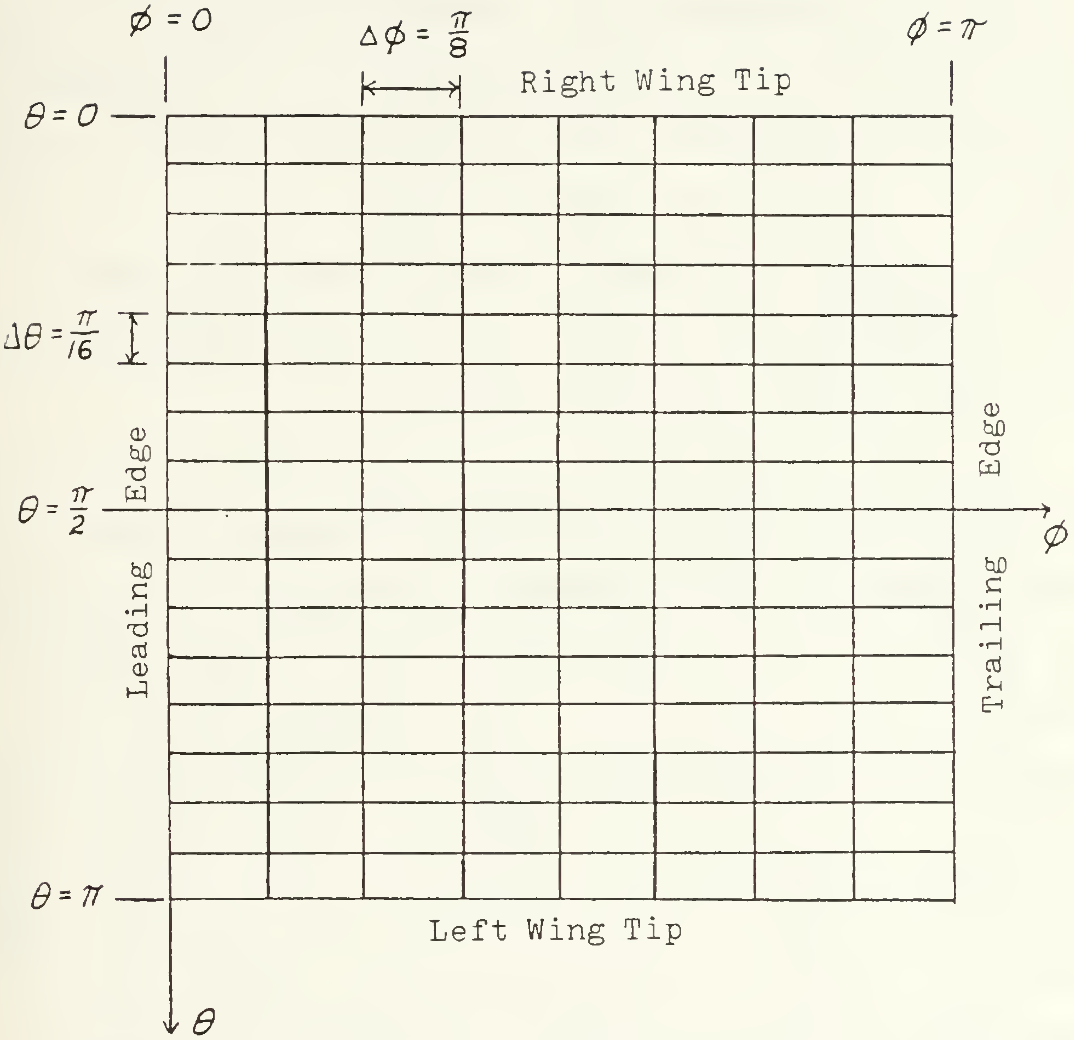


Figure 4. Computational Grid in (ϕ,θ) Plane

$$\theta_j = \Delta\theta(j-0.5); \quad \Delta\theta = 2\delta = \pi/16; \quad j = 1,2,\ldots,8 \qquad (6.2)$$

Also, along the leading and trailing edges of the wing

$$\phi_L(j) = 0; \quad \phi_t(j) = \pi \qquad (6.3)$$

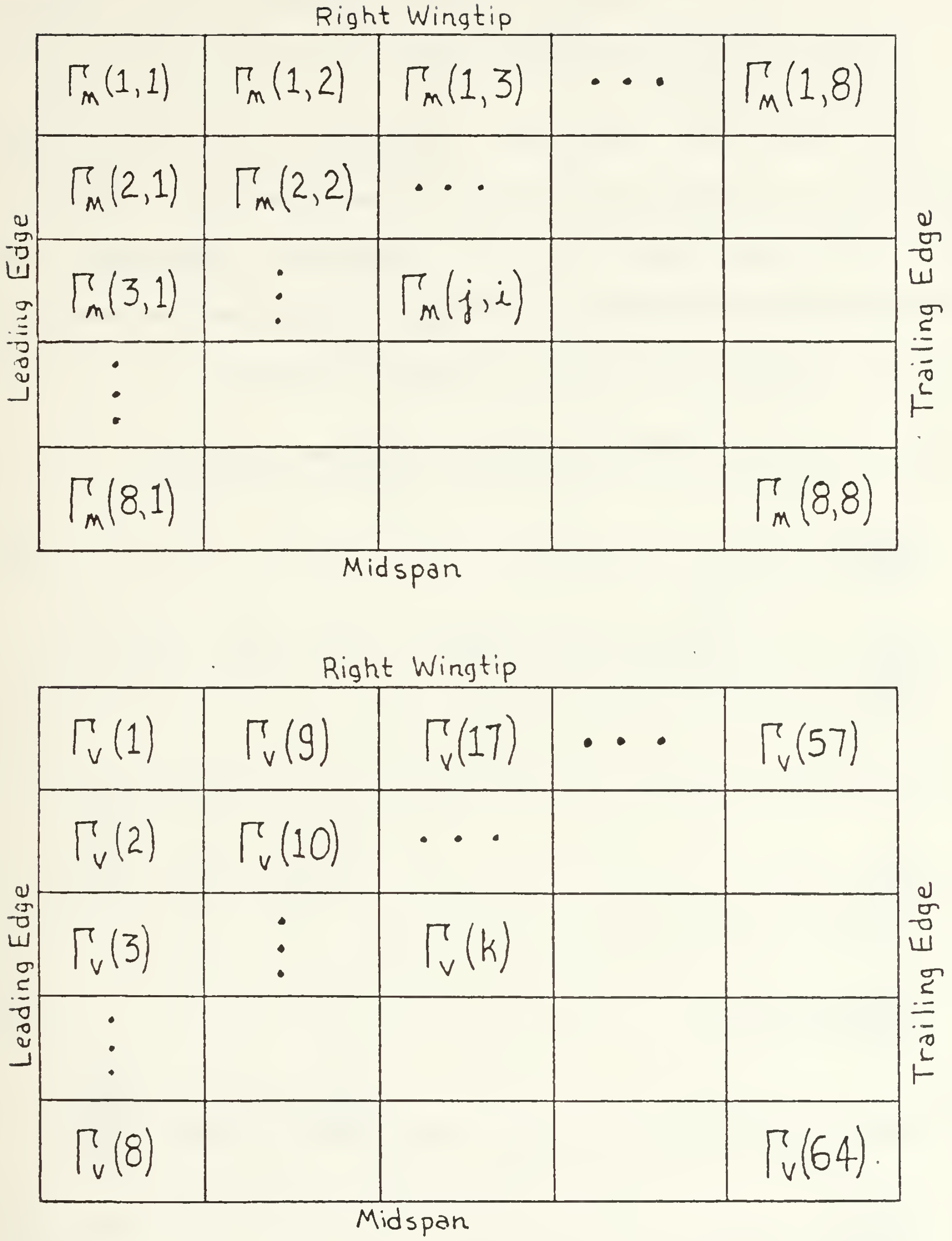
while along the wing tip and at midspan

$$\theta_t(i) = 0; \quad \theta_m(i) = \pi/2 \qquad (6.4)$$

The above equations will produce 64 grid points on the right semi-span with each point centered in its respective rectangular mesh block. The 'j' subscript varies spanwise from right wing tip to midspan while the 'i' subscript varies chordwise from leading edge to trailing edge. The unknown values of the circulation function at each of these grid points must be determined.

For every field element on the right wing there exists a corresponding mirror image field element on the left wing. By utilizing the symmetry between each field element and its mirror image, the range of the 'j' subscript can be restricted to 8 vice the 16 it would otherwise be. Fig. 5 depicts the planform distribution of the circulation function with appropriate subscripts for matrix and corresponding vector representation. All quantities with numerical values at respective grid points may be designated similarly. Should it become necessary to transform from matrix notation to vector notation or vice versa the following relationships are used, with 'k' denoting the respective vector subscript.

$$k = 8(i - 1) + j \text{ where } \Gamma_V(k) = \Gamma_M(j,i) \qquad (6.5)$$



Right Wingtip

Leading Edge

Trailing Edge

$\Gamma_m(1,1)$	$\Gamma_m(1,2)$	$\Gamma_m(1,3)$	$\cdots$	$\Gamma_m(1,8)$
$\Gamma_m(2,1)$	$\Gamma_m(2,2)$	$\cdots$		
$\Gamma_m(3,1)$	$\vdots$	$\Gamma_m(j,i)$		
$\vdots$				
$\Gamma_m(8,1)$				$\Gamma_m(8,8)$

Midspan

Right Wingtip

Leading Edge

Trailing Edge

$\Gamma_v(1)$	$\Gamma_v(9)$	$\Gamma_v(17)$	$\cdots$	$\Gamma_v(57)$
$\Gamma_v(2)$	$\Gamma_v(10)$	$\cdots$		
$\Gamma_v(3)$	$\vdots$	$\Gamma_v(k)$		
$\vdots$				
$\Gamma_v(8)$				$\Gamma_v(64)$

Midspan

Figure 5. Gamma Matrix and Vector Grids

$i = k/8 + 1$ (k/8 by integer division) (6.6)

$j = k - 8(i - 1)$ where $\Gamma_M(j,i) = \Gamma_v(k)$ (6.7)

B. VARIABLE NOTATION

The subscript notation introduced in the above paragraphs is used extensively in the subsequent analysis. The following notation for various functions and derivatives is also used. Note the different expressions that correspond to the same numerical value.

The first derivatives of the circulation function

$$\left(\frac{\partial\Gamma}{\partial\phi}\right) = \Gamma_\phi(j,i) = \Gamma_{\phi v}(k) \quad (6.8)$$

$$\left(\frac{\partial\Gamma}{\partial\theta}\right) = \Gamma_\theta(j,i) = \Gamma_{\theta v}(k) \quad (6.9)$$

The first derivatives at the leading and trailing edge

$$\left(\frac{\partial\Gamma}{\partial\phi}\right)_L = \Gamma_{\phi L}(j) \quad (6.10)$$

$$\left(\frac{\partial\Gamma}{\partial\phi}\right)_t = \Gamma_{\phi t}(j) \quad (6.11)$$

The second derivatives at the leading and trailing edge

$$\left(\frac{\partial^2\Gamma}{\partial\phi^2}\right)_L = \Gamma_{\phi\phi L}(j) \quad (6.12)$$

$$\left(\frac{\partial^2\Gamma}{\partial\phi^2}\right)_t = \Gamma_{\phi\phi t}(j) \quad (6.13)$$

The lift distribution is designated as follows

$$\Delta C_p)_{j,i} = P(j,i) = P_V(k) \qquad (6.14)$$

The lift distribution at the leading and trailing edge

$$\Delta C_p)_{Lj} = P_L(j) \qquad (6.15)$$

$$\Delta C_p)_{tj} = P_t(j) \qquad (6.16)$$

C. LIFT DISTRIBUTION

Consider Bernoulli's equation for the pressure above the wing and below the wing

$$p_u + (\rho/2)[V_\infty + \gamma_y/2]^2 = p_L + (\rho/2)[V_\infty - \gamma_y/2]^2 \qquad (6.17)$$

Rearranging and simplifying gives

$$\frac{(P_L - P_U)}{\frac{1}{2}\rho V_\infty^2} = \frac{\Delta p}{q} = \Delta C_p = 2\left(\frac{\gamma_y}{V_\infty}\right) = 2\,\hat{\gamma}_y = 2\left(\frac{\partial\hat{\Gamma}}{\partial x}\right) \qquad (6.18)$$

But,

$$\left(\frac{\partial\hat{\Gamma}}{\partial x}\right) = \frac{2}{AR}\,\frac{1}{c}\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right) \qquad (6.19)$$

and,

$$\left(\frac{\partial\tilde{\Gamma}}{\partial\xi}\right) = \frac{2}{\sin\phi}\left(\frac{\partial\Gamma}{\partial\phi}\right) \qquad (6.20)$$

Now, combining terms from above

$$\Delta C_p = 2\left(\frac{2}{AR}\right)\frac{1}{c}\left(\frac{2}{\sin\phi}\right)\left(\frac{\partial\Gamma}{\partial\phi}\right) = \frac{8}{AR}\,\frac{1}{c}\left(\frac{1}{\sin\phi}\right)\left(\frac{\partial\Gamma}{\partial\phi}\right) \qquad (6.21)$$

Or, in the notation introduced in section B

$$P(j,i) = \left(\frac{8}{AR}\right)\left\{\frac{\Gamma_\phi(j,i)}{c(j)\sin\phi_i}\right\} \qquad (6.22)$$

D. LEADING AND TRAILING EDGE REQUIREMENTS

The boundary conditions pertaining to equation (6.22) for the basic or ideal lift, and the additional lift, determine the nature of the circulation function and its derivatives. The Kutta condition fixes the pressure coefficient at zero at the trailing edge. For the additional lift, the pressure coefficient is infinite at the leading edge. However, for the basic lift, the pressure coefficient at the leading edge is finite. These specifications together with equation (6.21) or (6.22) determine the nature of the circulation function.

First, consider the Kutta condition which requires that the pressure coefficient be zero at the trailing edge. The denominator of equation (6.22) is also zero at the trailing edge, since $\phi = \pi$ and consequently, $\sin\phi = 0$. This means that Γ_ϕ must not be finite or the quotient becomes infinite. Therefore, Γ_ϕ at the trailing edge must be zero, resulting in an indeterminacy. Using L'Hopital's rule to determine the limit, which must be zero, the denominatior becomes $\cos\pi = -1$ and the numerator must be zero. This fixes the value of the second derivative of the circulation function with respect to ϕ at zero also. To summarize in equation form

$$\Delta C_p)_t = P_t(j) = 0 \qquad (6.23)$$

$$\left(\frac{\partial\Gamma}{\partial\phi}\right)_t = \Gamma_{\phi t}(j) = 0 \qquad (6.24)$$

$$\left(\frac{\partial^2\Gamma}{\partial\phi^2}\right)_t = \Gamma_{\phi\phi t}(j) = 0 \qquad (6.25)$$

Next, consider the condition that the pressure coefficient is infinite at the leading edge for the additional lift. At the leading edge, $\phi = 0$ and $\sin\phi = 0$; therefore, the denominator of equation (6.22) is again zero. Thus, Γ_ϕ is finite at the leading edge for the additional lift. Summarized in equation form

$$\Delta C_p)_L = P_L(j) = +\infty \qquad (6.26)$$

$$\left(\frac{\partial\Gamma}{\partial\phi}\right)_L = \Gamma_{\phi L}(j) > 0 \qquad (6.27)$$

Finally, for the basic lift the pressure coefficient is finite at the leading edge. A similar analysis to that for the trailing edge using L'Hopital's rule results in the second derivative of the circulation function with respect to ϕ being a finite value. Summarized in equation form

$$\Delta C_p)L = P_L(j) = (8/AR)\Gamma_{\phi\phi L}/c_j = \text{finite} \qquad (6.28)$$

$$\left(\frac{\partial\Gamma}{\partial\phi}\right)_L = \Gamma_{\phi L}(j) = 0 \qquad (6.29)$$

E. MIDSPAN AND WING TIP REQUIREMENTS

Along the right wing tip, denoted by subscript 'R', the boundary conditions are

$$\Gamma_R(i) = 0 \qquad (6.30)$$

$$\left(\frac{\partial \Gamma}{\partial \theta}\right)_R = \Gamma_R(i) = 0 \qquad (6.31)$$

At midspan,denoted by subscript 'm', allowance is made for a discontinuity in slope of the circulation function. Note that the circulation function is continuous at midspan, but its derivative is discontinuous at this location.

F. BASIC AND ADDITIONAL CIRCULATION

An understanding of the concepts of basic and additional circulation is necessary in the analysis of this problem. These concepts are briefly reviewed here.

Consider the circulation function $\Gamma(\phi,\theta)$ of a given wing at some arbitrary angle of attack α. This may be expressed in terms of a "basic" circulation function $\Gamma_B(\phi,\theta)$ and a "specific" additional circulation function $\Gamma_A(\phi,\theta)$ as follows

$$\Gamma(\phi,\theta) = \Gamma_B(\phi,\theta) + (\alpha - \alpha_B)\Gamma_A(\phi,\theta) \qquad (6.32)$$

where $(\alpha - \alpha_B)$ is termed the additional angle of attack.

It is useful to designate the product $(\alpha - \alpha_B)\Gamma_A(\phi,\theta)$ as the "net" additional circulation and to designate the quantity $\Gamma_A(\phi,\theta)$ itself as the "specific" additional circulation.

Differentiating equation (6.32) with respect to α gives

$$\frac{d}{d\alpha}[\Gamma(\phi,\theta)] = 0 + \Gamma_A(\phi,\theta) \tag{6.33}$$

Notice that $\Gamma_A(\phi,\theta)$ has the character of circulation per radian, and that it is numerically equal to the value which the net circulation would reach if linearly extrapolated to an additional angle of attack of one radian. In fact, the additional angle of attack $(\alpha - \alpha_B)$ is restricted to small values; therefore, the net additional circulation remains small in comparison with the specific additional circulation.

All other aerodynamic parameters which are proportional to circulation follow a linear law analogous to that stated in equations (6.32) and (6.33). For example, the wing lift coefficient C_L conforms to the expressions

$$C_L = C_{LB} + (\alpha - \alpha_B)C_{LA} \tag{6.34}$$

and

$$\frac{dC_L}{d\alpha} = 0 + C_{LA} \tag{6.35}$$

Again note that the specific additional lift coefficient C_{LA} is identical with the lift curve slope $(dC_L/d\alpha)$ and is numerically equal to the value that the net additional lift would reach if linearly extrapolated to an additional angle of attack of one radian. Once again, however, the additional angle of attack $(\alpha - \alpha_B)$ is restricted to small values.

Basic and additional lift are also discussed in Ref. 3.

VII. GENERAL PROBLEM ANALYSIS

A. PROBLEM SPECIFICATION

As mentioned previously, the calculation sequence may involve determining the wing slope function from a specified basic lift distribution, or determining the lift distribution from a specified wing slope function. Of interest here are both the basic and additional lift distributions.

The values on the right wing span are utilized in all calculations. The right wing span is divided into eight chordwise and eight spanwise stations, resulting in sixty-four control points over the semi-span. To satisfy the condition of no flow through the wing surface, the induced velocity must balance the normal component of the free stream velocity, or

$$w_p + V_\infty \sin \alpha = \partial z / \partial x \qquad (7.1)$$

Setting the free stream velocity to one unit and approximating $\sin \alpha$ with α for small angles of attack, gives

$$(\partial z / \partial x) - \alpha = w_p = w_p{}' + w_p{}'' \qquad (7.2)$$

By analogy with equation (6.32), the overall wing slope function at each control point may be represented as the sum of two components associated with the basic lift and the additional lift, respectively. Thus

$$w_p = w_{pB} + (\alpha - \alpha_B) w_{pA} \qquad (7.3)$$

where w_{pB} reflects the slope of the wing camber surface according to the relation

$$w_{pB} = (\partial z/\partial x) - \alpha_B \qquad (7.4)$$

while

$$w_{pA} = -1 \qquad (7.5)$$

reflects the uniform change in wing slope per radian of additional angle of attack.

Note that the right hand side of equation (7.2) consists of the sum of equations (4.24) and (4.39). When solving for the additional lift, the only unknowns in equations (4.24) and (4.39) are the derivatives of the circulation function. These derivatives are expressed in terms of the initially unknown values of the circulation. With sixty-four unknown values of circulation, sixty-four equations are required for a solution. Each control point generates an equation involving terms of the unknown circulation distribution. This produces sixty-four equations in sixty-four unknowns which must be solved simultaneously to determine the circulation values. This solution involves the inversion of a 64 by 64 matrix.

Expressions for the partial derivatives of the circulation function in terms of the unknown circulation values are needed and are derived below.

When solving the basic lift problem with a specified pressure distribution, the unknown is the wing slope function, w_p. This solution involves multiplication by a 64 by 64 matrix, but inversion of the matrix is not required.

B. CHORDWISE DERIVATIVES OF CIRCULATION

In developing finite difference equations, a polynomial is passed through a number of points in the particular area of interest. The polynomial used in each case must be of sufficient order to ensure that the required derivatives exist, and it must satisfy the boundary conditions. The finite difference equations utilized for the central difference formulas require a point on either side of the point in question. Hence, the central difference equations are valid for all mesh points except those that lie adjacent to the leading and trailing edges. For the leading edge, let

$$\Gamma = A_1\phi + (1/2)A_2\phi^2 + O[\epsilon^3] \qquad (7.6)$$

Differentiating,

$$\Gamma_\phi = A_1 + A_2\phi + O[\epsilon^2] \qquad (7.7)$$

Equations (7.6) and (7.7) satisfy the boundary conditions for the additional lift. These are that Γ vanish and Γ_ϕ is finite at the leading edge where $\phi = 0$. Applying equation (7.6) at two grid points adjacent to the leading edge, solving for A_1 and A_2, and substituting into equation (7.7), a difference expression for Γ_ϕ in terms of Γ_1 and Γ_2 is obtained as follows

$$\Gamma_\phi = (1/\Delta\phi)[\Gamma_1 + (1/3)\Gamma_2] \qquad (7.8)$$

Therefore, for any mesh point adjacent to the leading edge, the following subscript notation is used for Γ_ϕ

$$\Gamma_\phi(j,1) = (1/\Delta\phi)[\Gamma(j,1) + (1/3)\Gamma(j,2)] \qquad (7.9)$$

A similar analysis applied to the basic lift results in the following

$$\Gamma_\phi(j,1) = (1/\Delta\phi)[-(\Delta\phi/2)^2\Gamma_{\phi\phi_L}(j) + 6\Gamma(j,1)] \qquad (7.10)$$

The procedure for determining the difference formula applicable near the trailing edge requires defining a function for the circulation that is numerically equal to Γ_t at the trailing edge with first and second derivatives equal to zero. This requires beginning with a cubic equation; therefore, let

$$\Gamma = \Gamma_t + (1/6)A_3(\pi - \phi)^3 \qquad (7.11)$$

Differentiating once,

$$\Gamma_\phi = -(1/2)A_3(\pi - \phi)^2 \qquad (7.12)$$

Differentiating again,

$$\Gamma_{\phi\phi} = +A_3(\pi - \phi) \qquad (7.13)$$

Applying equation (7.11) at two trailing grid points, solving for A_3, and substituting into (7.12), a difference expression for Γ_ϕ in terms of $\Gamma(j,7)$ and $\Gamma(j,8)$ is obtained as follows

$$\Gamma_\phi(j,8) = (1/\Delta\phi)[(-3/13)\Gamma(j,7) + (3/13)\Gamma(j,8)] \qquad (7.14)$$

Also

$$\Gamma_t(j) = (-1/26)\Gamma(j,7) + (27/26)\Gamma(j,8) \qquad (7.15)$$

For mesh points 2 through 7 in the chordwise direction, central differences are used which results in the following

$$\Gamma_\phi(j,i) = (1/\Delta\phi)[(-1/2)\Gamma(j,i-1) + (1/2)\Gamma(j,i+1)] \qquad (7.16)$$

The difference equations for the trailing edge and the central points are also valid for the basic lift problem.

C. SPANWISE DERIVATIVES OF CIRCULATION

Along the right wing tip, the circulation is zero; however, the first derivative is finite. These conditions are precisely analogous to the boundary conditions along the leading edge for the additional lift. Hence, the same difference equation developed above for these boundary conditions may be used in this case. For the right wing tip mesh points

$$\Gamma_\theta(1,i) = (1/\Delta\theta)[\Gamma(1,i) + (1/3)\Gamma(2,i)] \tag{7.17}$$

Central difference formulas are also the same

$$\Gamma_\theta(j,i) = (1/\Delta\theta)[(-1/2)\Gamma(j-1,i) + (1/2)\Gamma(j+1,i)] \tag{7.18}$$

At midspan, allowance is made for a discontinuity in the slope of the circulation function as mentioned previously. Again, a polynomial suitable to satisfy the boundary conditions is assumed; therefore, let

$$(\Gamma - \Gamma_8) = A_1(\theta - \theta_8) + (1/2)A_2(\theta - \theta_8)^2 \tag{7.19}$$

and

$$\Gamma_\theta = A_1 + A_2(\theta - \theta_8) \tag{7.20}$$

Satisfying equation (7.19) at j = 6 and j = 7, and solving for A_1 and A_2, gives the needed coefficients in equations (7.19) and (7.20). Now, applying equation (7.19) at midspan and equation (7.20) at midspan and at j = 8 gives

$$\Gamma_m(i) = (3/8)\Gamma(6,i) - (5/4)\Gamma(7,i) + (15/8)\Gamma(8,i) \tag{7.21}$$

$$\Gamma_{\theta m}(i) = (1/\Delta\theta)[\Gamma(6,i) - 3\Gamma(7,i) + 2\Gamma(8,i)] \tag{7.22}$$

$$\Gamma_\theta(8,i) = (1/\Delta\theta)[(1/2)\Gamma(6,i)-2\Gamma(7,i)+(3/2)\Gamma(8,i)] \tag{7.23}$$

D. FIELD ELEMENT AND IMAGE ELEMENT

Recall that the only differences in the values over the right and left semi-span are the algebraic signs of η, Γ_θ, and σ. Therefore, w_p' may be thought of as having two parts, one part consisting of the contribution that a particular field point on the right semi-span makes on a specified control point, and another part consisting of the contribution that the image field point on the left semi-span makes on the same control point. The same concept applies to w_p'', the trailing vorticity integral along the trailing edge. Equations (4.24) and (4.39) are modified slightly so that the effect of a field point on the right semi-span and the effect of its image on the left semi-span may be calculated simultaneously, using symmetry relations. Let the subscript '1' represent the right semi-span and let subscript '2' represent the left semi-span. Making the substitutions for the symmetric elements and collecting terms, equations (4.24) and (4.39) may be rewritten as

$$w_p' = \frac{1}{4\pi}\left(\frac{2}{AR}\right)\int\!\!\int\left\{\left[\frac{F_1}{r_1^3}+\frac{F_2}{r_2^3}\right]\sin\theta\,\Gamma_\phi-\left[\frac{(\eta-\eta_p)}{r_1^3}+\frac{(\eta+\eta_p)}{r_2^3}\right]\frac{c}{2}\sin\phi\,\Gamma_\theta\right\}d\phi\,d\theta \qquad (7.24)$$

$$w_p'' = -\frac{1}{4\pi}\left(\frac{2}{AR}\right)\int\left[\frac{G_1}{(\eta-\eta_p)}+\frac{G_2}{(\eta+\eta_p)}\right]\Gamma_{t\theta}\,d\theta \qquad (7.25)$$

Equations (7.2), (7.24) and (7.25), when evaluated at each control point over the semi-span, produce sixty-four equations in the sixty-four unknown values of the circulation function. The solution to this system of equations produces the desired values of the circulation function at the mesh points over the right semi-span.

E. BASIC LIFT

In the basic lift problem a pressure distribution is specified over the wing, leaving the corresponding wing slope function to be determined. In this case, equation (6.22), rearranged, gives the chordwise derivatives of the circulation function at each mesh point on the wing, as follows

$$\Gamma_{\phi}(j,i) = \frac{AR}{8} c(j) \sin\phi_i \; P(j,i) \qquad (7.26)$$

At the leading edge, equation (6.28) applies

$$\Gamma_{\phi\phi_L}(j) = \frac{AR}{8} c(j) \; P_L(j) \qquad (7.27)$$

Now, equations (7.10), (7.14), and (7.16) are expressions for Γ_{ϕ} in terms of the unknown circulation values. These equations are repeated below with known values collected on the left side of each equation.

$$\Gamma_{\phi}(j,1) + (\Delta\phi/4)\Gamma_{\phi\phi_L}(j) = (1/\Delta\phi)6\Gamma(j,1) \qquad (7.10)$$

$$\Gamma_{\phi}(j,i) = (1/\Delta\phi)[(-1/2)\Gamma(j,i-1) + (1/2)\Gamma(j,i+1)] \tag{7.16}$$

$$\Gamma_{\phi}(j,8) = (1/\Delta\phi)[(-3/13)\Gamma(j,7) + (3/13)\Gamma(j,8)] \tag{7.14}$$

All values on the left hand sides of the above three equations are determined by equations (7.26) and (7.27). The solution for the circulation values requires inversion of a 8 by 8 matrix.

Now, equations (7.17), (7.18), and (7.15) are utilized to calculate the spanwise derivatives of the circulation function. All variables on the right hand sides of equations (7.24) and (7.25) are now known. The wing slope function may be written as

$$w_p(k_p) = \sum_{k} B(k_p,k)\Gamma(k) \tag{7.28}$$

where B is a 64 by 64 matrix of influence coefficients whose exact definition is implied by equations (7.24) and (7.25). A more detailed analysis of matrix B is given in Ref. 4.

F. ADDITIONAL LIFT

In the additional lift problem, $w_p = -1$, and the pressure distribution over the wing is desired. The following previously developed equations are required.

$$\Gamma_{\phi}(j,1) = (1/\Delta\phi)[\Gamma(j,1) + (1/3)\Gamma(j,2)] \tag{7.9}$$

$$\Gamma_{\phi}(j,i) = (1/\Delta\phi)[(-1/2)\Gamma(j,i-1) + (1/2)\Gamma(j,i+1)] \tag{7.16}$$

$$\Gamma_{\phi}(j,8) = (1/\Delta\phi)[(-3/13)\Gamma(j,7) + (3/13)\Gamma(j,8)] \tag{7.14}$$

$$\Gamma_{\theta}(1,i) = (1/\Delta\theta)[\Gamma(1,i) + (1/3)\Gamma(2,i)] \tag{7.17}$$

$$\Gamma_{\theta}(j,i) = (1/\Delta\theta)[(-1/2)\Gamma(j-1,i) + (1/2)\Gamma(j+1,i)] \tag{7.18}$$

$$\Gamma_{\theta}(8,i) = (1/\Delta\theta)[(1/2)\Gamma(6,i)-2\Gamma(7,i)+(3/2)\Gamma(8,i)] \tag{7.23}$$

$$\Gamma_{t}(j) = (-1/26)\Gamma(j,7) + (27/26)\Gamma(j,8) \tag{7.15}$$

The wing slope function in this case may be written as

$$w_p(k_p) = \sum_{k} A(k_p,k)\Gamma(k) \tag{7.34}$$

where A is a 64 by 64 matrix of influence coefficients whose exact definition is again implied by equations (7.24) and (7.25). Matrix A must be inverted in this case to determine the circulation function.

Once the circulation function is known, its derivative with respect to ϕ is found by substituting the known values into equations (7.9), (7.14), and (7.16). Equation (6.26) is then utilized to determine the pressure distribution. Other aerodynamic parameters may then be readily calculated for the wing in question.

The matrices and calculation procedures presented in the above paragraphs are described in detail in a concurrent analysis conducted by E. M. Barber [Ref. 4].

VIII. SUMMARY

The solution of the system of equations developed in this analysis is best suited for processing by matrix methods on a digital computer. A brief summary of the procedural steps required in the calculations for the additional lift and the basic lift are given in the following paragraphs.

A. ADDITIONAL LIFT

Given $w_p = -1$:

First, calculate matrix A utilizing equations (7.9), (7.14), (7.15), (7.16), (7.17), (7.18), and (7.23) together with the planform parameters in equations (7.24) and (7.25).

Next, invert matrix A and multiply by vector w_p to determine the circulation values.

Now, determine the chordwise derivative of the circulation function utilizing equations (7.9), (7.14), and (7.16).

Finally, use equation (6.22) to calculate the pressure distribution.

B. BASIC LIFT

Given P(j,i):

First, calculate the chordwise derivative of the circulation function using equation (6.22), and the second derivative factor using equation (6.28).

Next, compute the circulation function using equations (7.9), (7.14), and (7.16).

Now, compute matrix B.

Finally, multiply matrix B by the circulation vector to obtain w_p.

IX. CONCLUSIONS

Although no numerical results are yet available, there are ample grounds for confidence in the validity of the calculation method presented in this thesis. All derivations are based on sound mathematical procedures and assumptions. This analysis, along with the parallel effort presented in Ref. 4, has satisfied the initial purpose of the overall research effort, which was to develop an adequate mathematical model for calculating the aerodynamic performance of a wing of arbitrary planform in inviscid, incompressible flow.

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